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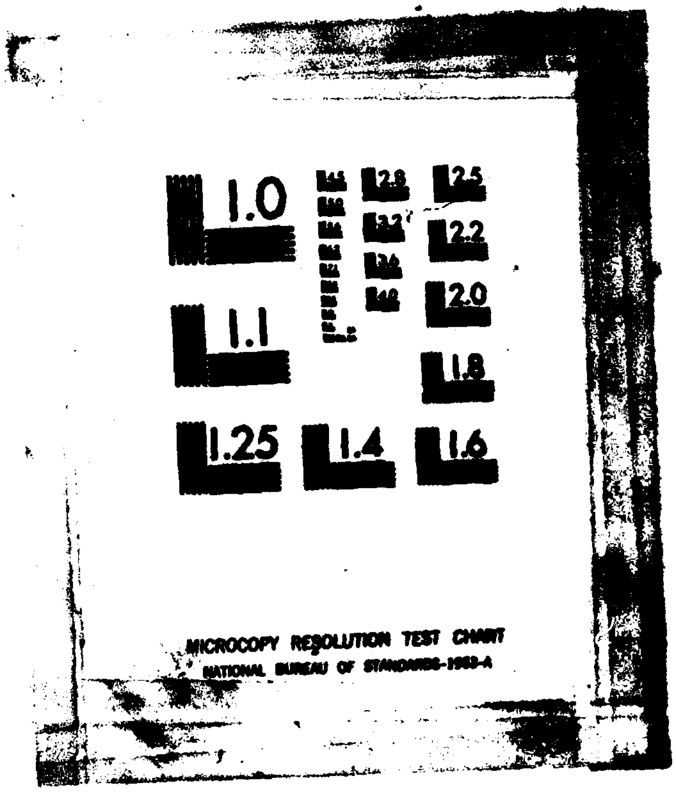
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On the Construction of a Modulating Multiphase Wavetrain for a Perturbed KdV Equation

by

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This paper summarizes the status of a direct construction of an asymptotic representation of a modulating multiphase wavetrain for a class of perturbed KdV equations. This class includes the KdV-Burgers' equation. The calculations apply on a boundary between dispersive and dissipative behavior. The construction proceeds by standard asymptotic methods. The result of the construction is an invariant representation of the reduced equations which

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permits their diagonalization. While mathematically the construction is incomplete, care is taken to correctly identify the mathematical status of each step in the construction. The equivalence of this constructive approach with the postulated averaging of conservation laws is established for two phase waves.

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Abstract

This paper summarizes the status of a direct construction of an asymptotic representation of a modulating multiphase wavetrain for a class of perturbed KdV equations. This class includes the KdV-Burgers' equation. The calculations apply on a boundary between dispersive and dissipative behavior. The construction proceeds by standard asymptotic methods. The result of the construction is an invariant representation of the reduced equations which permits their diagonalization. While mathematically the construction is incomplete, care is taken to correctly identify the mathematical status of each step in the construction. The equivalence of this constructive approach with the postulated averaging of conservation laws is established for two phase waves.

I. Introduction

In this paper we study the initial value problem for a perturbed Korteweg de Vries (KdV) equation:

$$U_t - 6UU_x + \varepsilon^2 U_{xxx} + \beta f(U, \varepsilon U_x, \varepsilon^2 U_{xx}, \ldots) = 0$$

$$U(x,t=0) = U_{in}\left(\frac{x}{\varepsilon}; x\right).$$

Here ε denotes the small parameter $(0 < \varepsilon << 1)$, while $\beta \geq 0$ is a constant of order unity. The initial data U_{in} depends upon two spatial scales, a fast scale x/ε and a slower scale x. The reason we tie the fast scale in the initial data to the small parameter ε in the equation is that the KdV equation $(\beta = 0)$ in the "small dispersion limit" $(0 < \varepsilon << 1)$ is known to develop oscillations whose spatial wavelength is $O(\varepsilon)$. Actually, we will restrict our attention to a special class of initial data - a slowly varying N-phase waveform - which we will describe in detail later.

Our goal is to construct a representation of the solution $U = U^{(\varepsilon)}(x,t)$ which is valid for small, but finite, ε.

There are two general reasons for our interest in this problem. The first is technical. We desire to develop a method for the construction of $U^{(\varepsilon)}$ which (i) is general enough to include external perturbations (such as dissipation) of a completely integrable system, (ii) shows the validity of our prescription [1,2,3] of "averaged conservation laws" for the description of modulating N phase wavetrains, (iii) is a sufficiently standard mathematical technique that the construction can provide a first step toward a rigorous proof of the validity of the representation. There are cases (the sine-Gordon equation, [4,5] for example) where the wave train is modulationally unstable. To study this instability one must retain additional terms in the modulation equations. The technique we are developing is sufficiently standard that a systematic retention of such higher order terms can be attempted.

The second reason we are interested in this problem concerns the Korteweg-de Vries - Burgers' equation itself. Consider the special perturbation $f = - \varepsilon^2 U_{xx}$. Then the equation becomes

$$U_t - 6\, UU_x + \varepsilon^2 U_{xxx} - \delta\, U_{xx} = 0$$

with $\delta = \beta \varepsilon^2$. Relax this relation between δ and ε for the moment and think of the equation as depending upon two independent small parameters, ε and δ . When the dispersive term is absent ($\varepsilon \equiv 0$), the solution U tends, as $\delta \downarrow 0$, to a sequence of shocks whose speeds are fixed by the entropy condition. On the other hand, when the dissipative term is absent ($\delta = 0$), the solution U becomes very oscillatory for small ε . In this case the weak limit of $U^{(\varepsilon)}$ as $\varepsilon \downarrow 0$ is described by a family of nonlinear, hyperbolic equations [6,7,8,9]. When both the dissipative and dispersive terms are present, but small ($0 < \varepsilon, \delta < < 1$), shocks will describe the weak limit provided the dissipation dominates. Recently it has been established [10] that shocks will describe the weak limit as $\varepsilon \downarrow 0$, $\delta \downarrow 0$, provided the dissipation parameter $\delta \geq \beta \varepsilon$. Our calculations ($\delta = \beta \varepsilon^2$) describe the boundary between dissipative and dispersive behavior. For stronger dissipation, shocks apply; on this boundary, the behavior for small ε is described by the equations derived in this paper; for still weaker dissipation, the weak limit is described by [6,7,8,9].

The method that we are developing has several main advantages: (i) It constructs a representation of the wave which, since it is valid for finite, but small ε, retains the oscillatory structure of the wave. (ii) It extends naturally to small perturbations of integrable systems.

This work has two shortcomings. The first concerns a limitation of the method itself. It demands a very restricted class of initial data-slowly varying N-phase waves. This means that, for general data, it is not valid uniformly in space; rather, it applies only in the vicinity of the "shock front". The second shortcoming concerns the work reported herein; not the method itself. This work, although systematic, is not rigorous; we view it as the first step toward a rigorous derivation. In the text we will clearly identify those points where rigor is absent.

This work continues our study of the modulation of completely integrable wave trains [1,2,3,4,5]. The procedure of averaging conservation laws to obtain modulation equations, as well as placing these equations in Riemann invariant form, was initiated in [11,12] for the single phase case. The constructive method employed here follows single phase work in [13,14] and the multi-phase procedure of [15]. In the single phase case the results are not new, although our arguments are somewhat more systematic; in the multiphase case, our results are new because of our use of the completely integrable exact theory. In the single phase case, dissipative perturbations of nonlinear, dispersive wave trains have been analysed previously [12,16,17]. The (weak) zero dispersion limit of the KdV equation is studied in [8,9,10].

The last section of this paper is motivated by the rigorous work of [19,20,10].

Finally, I acknowledge many conversations with R. DiPerna, H. Flaschka, M. G. Forest, C. D. Levermore, H. McKean, G. Papanicolaou and S. Venekidas. My work has certainly benefited from each of these interactions.

II. Definition of the Problem

Consider the initial value problem

$$U_t - 6\,UU_x + \varepsilon^2\,U_{xxx} + \beta\,f(U,\,\varepsilon U_x,\,\varepsilon U_t,\,\varepsilon^2 U_{xx},\ldots) = 0\,,$$

$$U(x,t=0) = W_{in}^{(N)}\left(\frac{x}{\varepsilon}\,;\,\vec{\lambda}(x)\right)\,. \tag{II.1a,b}$$

Here $0 < \varepsilon << 1$ and $\beta = O(1)$. The initial data is a slowly modulating N-phase waveform for the KdV equation which we now describe.

The KdV equation has a family of exact solutions of the form

$$U(x,t) = W_N\left(\frac{\Theta_1(x,t)}{\varepsilon},\ldots,\frac{\Theta_N(x,t)}{\varepsilon}\,;\,\lambda_0,\ldots,\lambda_{2N}\right) \tag{II.2a}$$

where the N "phases" $\Theta_1,\ldots,\Theta_N$ depend linearly upon x and t ,

$$\Theta_j(x,t) \equiv \kappa_j\,x - \omega_j\,t\,. \tag{II.2b}$$

The waveform W_N is 2π periodic in each argument $\theta_j \equiv \Theta_j/\varepsilon$. The family of "N phase waves" (II.2a) is indexed by 2N+1 parameters $\vec{\lambda} = (\lambda_0,\,\lambda_1,\ldots,\lambda_{2N})$, which fix the N spatial wavenumbers $\kappa = (\kappa_1,\ldots,\kappa_N)$ and the N temporal frequencies $\omega = (\omega_1,\ldots,\omega_N)$. The explicit formulas for $\kappa = \kappa(\vec{\lambda})$ and $\omega = \omega(\vec{\lambda})$, as well as more details about these N phase waves, will be given later. For now, we only remark that in the single phase $(N=1)$ case, (II.2) is a 2π periodic traveling wave solution of the KdV equatiom.

The initial data (II.1b) is described in terms of an N phase wave W_N as follows: Prescribe smooth functions $\vec{\lambda} = \vec{\lambda}(x)$ in terms of which

$$W_{in}^{(N)}\left(\frac{x}{\varepsilon};\ \vec{\lambda}(x)\right) \equiv W_N\left(\frac{\Theta_1(x)}{\varepsilon},\ldots,\frac{\Theta_N(x)}{\varepsilon};\ \vec{\lambda}(x)\right) \qquad \text{(II.3a,b)}$$

where

$$\partial_x \Theta_j \equiv \kappa_j(\vec{\lambda}(x)) \quad , \quad \vec{\lambda}(x) \ \text{ prescribed}$$

Our problem is to construct a representation of $U = U^{\varepsilon}$ which is valid for small ε. We use a standard asymptotic method which begins with an ansatz:

$$U \sim U^{\varepsilon}\left[\frac{\Theta_1^{\varepsilon}(x,t)}{\varepsilon},\ldots,\frac{\Theta_N^{\varepsilon}(x,t)}{\varepsilon};\ x,t\right] . \qquad \text{(II.4)}$$

Here the (x,t) dependence of $\Theta^{\varepsilon} = (\Theta_1^{\varepsilon},\ldots,\Theta_N^{\varepsilon})$ is to be determined. We define

$$\begin{aligned} \kappa^{\varepsilon}(x,t) &\equiv \partial_x \Theta^{\varepsilon}(x,t) \\ \omega^{\varepsilon}(x,t) &\equiv \partial_t \Theta^{\varepsilon}(x,t) \quad , \end{aligned} \qquad \text{(II.5)}$$

and note that

$$\partial_t\, U^{\varepsilon} \to \left(\frac{\omega^{\varepsilon}\cdot\nabla}{\varepsilon} + \partial_t\right)U^{\varepsilon}$$

$$\partial_x\, U^{\varepsilon} \to \left(\frac{\kappa^{\varepsilon}}{\varepsilon}\cdot\nabla + \partial_x\right)U^{\varepsilon} \ ,$$

where $\nabla U^{\varepsilon} \equiv (\frac{\partial}{\partial\theta_1} U^{\varepsilon}, \ldots, \frac{\partial}{\partial\theta_N} U^{\varepsilon}) = \varepsilon\, (\frac{\partial}{\partial\theta_1} U^{\varepsilon}, \ldots, \frac{\partial}{\partial\theta_N} U^{\varepsilon})$. Of course, definition (II.5) implies N consistency (or integrability) conditions for θ_j:

$$\partial_t \kappa^{\varepsilon} = \partial_x \omega^{\varepsilon} \ . \tag{II.6}$$

In terms of this ansatz, equation (II.1a) becomes

$$\frac{1}{\varepsilon}\left[\omega^{\varepsilon}\cdot\nabla U^{\varepsilon} - 6U^{\varepsilon}\, \kappa^{\varepsilon}\cdot\nabla U^{\varepsilon} + (\kappa^{\varepsilon}\cdot\nabla)^3 U^{\varepsilon}\right]$$

$$+ \left[U_t^{\varepsilon} + 6U^{\varepsilon}U_x^{\varepsilon} + 3\left((\kappa^{\varepsilon}\cdot\nabla)^2\, U_x^{\varepsilon} + (\kappa^{\varepsilon}\cdot\nabla)(\kappa_x^{\varepsilon}\cdot\nabla)U^{\varepsilon}\right) + \beta f(U^{\varepsilon}, (\kappa^{\varepsilon}\cdot\nabla+\varepsilon\partial_x)U^{\varepsilon}, \ldots)\right]$$

$$+ \ \varepsilon\left[\kappa^{\varepsilon}\cdot\nabla U_{xx}^{\varepsilon} + 3\ \kappa_x^{\varepsilon}\cdot\nabla U_x^{\varepsilon} + \kappa_{xx}^{\varepsilon}\cdot\nabla U^{\varepsilon}\right] + \varepsilon^2 \left[U_{xxx}^{\varepsilon}\right] = 0 \ .$$

In this equation, expansions of the form

$$\begin{aligned} U^{\varepsilon} &\sim W + \varepsilon\, U^{(1)} + \varepsilon^2\, U^{(2)} + \ldots \\ \kappa^{\varepsilon} &\sim \kappa + \varepsilon\, \kappa^{(1)} + \varepsilon^2\, \kappa^{(2)} + \ldots \\ \omega^{\varepsilon} &\sim \omega + \varepsilon\, \omega^{(1)} + \varepsilon^2\, \omega^{(2)} + \ldots \end{aligned} \tag{II.7}$$

lead to the following sequence of problems:

$$O(\varepsilon^{-1}): \quad (\omega\cdot\nabla)\, W - 6\, W(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3\, W = 0 \tag{II.8a}$$

$$O(\varepsilon^{j}): \qquad L\, U^{(j)} + F^{(j)} = 0\ , \quad j = 0,1,2,\ldots \tag{II.8b}$$

where the linear operator L is given by

$$L \equiv \omega\cdot\nabla - 6\ (\kappa\cdot\nabla)\ W + (\kappa\cdot\nabla)^3\ . \tag{II.9}$$

The inhomogeneity $F^{(0)}$ is given by

$$F^{(0)} = \left[W_t - 6WW_x + 3\left\{(\kappa\cdot\nabla)^2 W_x + (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W\right\} + \beta\ f(W, (\kappa\cdot\nabla)W, \ldots)\right]$$

$$+ [\omega^{(1)}\cdot\nabla W - 6\ W\ \kappa^{(1)}\cdot\nabla W + 3(\kappa\cdot\nabla)^2\ (\kappa^{(1)}\cdot\nabla)\ W]\ , \tag{II.10}$$

with similar, but more complicated, formulas for $F^{(j)}$, $j \geq 1$. The sequence of problems (II.8) must now be studied; however, we first summarize some background material from the theory of the inverse spectral representation of KdV. This material will be used throughout the paper.

III. Inverse Spectral Theory for the KdV Equation

The background material in this section may be found in [1,21] which contain references to the original literature. Let $q = q(y,\tau)$ satisfy the KdV equation,

$$q_\tau = 6\, qq_y - q_{yyy} \,. \tag{III.1}$$

By considering the "Lax representation" of this equation, one realizes that it arises as the integrability condition for the linear system

$$\begin{aligned} &[-\,\partial_{yyy} + 2(q\partial_y + \partial_y q)]\ \Psi = 4\lambda\ \partial_y \Psi \\ &\partial_\tau\ \Psi = -\,2q_y\Psi + 2(q+2\lambda)\partial_y\Psi \quad . \end{aligned} \tag{III.2a,b}$$

The function Ψ is known as a "squared eigenfunction". Equation (III.2a) is an eigenvalue problem for the squared eigenfunction Ψ; equation (III.2b) defines its time flow. The pair (III.2a,b) is compatible since q satisfies KdV.

The pair of equations (III.2) is fundamental in the theory of the KdV equation. Here we use the pair (i) to generate an infinite family of conservation laws, (ii) to provide a representation of the N-phase wave solutions of (II.8a), (iii) to provide formulas for the fluxes and densities of the conservation laws in terms of the N-phase waves, and (iv) to compute necessary averages.

III.A. Fundamental Conservation Law

Let q solve KdV and Ψ solve the pair (III.2). Then it immediately follows that Ψ satisfies the conservation law

$$\partial_\tau[\Psi] + \partial_y\,[6(q-2\lambda)\Psi - 2\partial_{yy}\Psi] = 0 \quad . \tag{III .3}$$

This fundamental conservation law generates an infinite family as follows: One seeks a solution of (III.2a) which has the asymptotic behavior

$$\Psi(y,\tau;\lambda) \simeq \frac{1}{\sqrt{2\lambda}} + \sum_{j=1}^{\infty} \Psi_j(y,\tau)(2\lambda)^{-j-1/2} \quad \text{as} \quad \lambda \to \infty. \tag{III.4}$$

This ansatz in (III.2a) leads to a recursion relation for the coefficients $\Psi_j(y,\tau)$,

$$\partial_y\,\Psi_{j+1} = [-\tfrac{1}{2}\,\partial_{yyy} + q\partial_y + \partial_y\,q]\Psi_j\ ,\ i = 1,2,\ldots \tag{III.5}$$

$$\Psi_0 = 1\ .$$

Recursion relation (III.5), together with certain choices of integration constants, provides formulas for Ψ_j in terms of q and its derivatives. We list the first few:

$$\Psi_0 = 1$$
$$\Psi_1 = q$$
$$\Psi_2 = \tfrac{1}{2}\,(3q^2-q_{yy})$$
$$\Psi_3 = \tfrac{5}{2}\,q^3 + \tfrac{5}{4}\,q_y^2 - \partial_y(\tfrac{5}{2}\,qq_y - \tfrac{1}{4}\,q_{yyy})$$

$$\Psi_4 = \frac{35}{8} q^4 + \frac{35}{4} q q_y^2 + \frac{7}{8} q_{yy}^2 + \partial_y \Big[- 5q^2 q_y + \frac{1}{2} q q_{yyy} - \frac{3}{4} q_y q_{yy} - \frac{1}{2} \partial_y \Psi_3 \Big]$$

By inserting the asymptotic behavior (III.4) into the fundamental conservation laws (III.3), we obtain a family of conservation laws for the KdV equation:

$$\partial_\tau [\Psi_j] + \partial_y [6(q\Psi_j - \Psi_{j+1}) - 2\partial_{yy}\Psi_j] = 0 \ , \ j = 1,2,\ldots \qquad \text{(III.7)}$$

III. B. <u>Solutions of the Adjoint Linearized KdV Equation</u>

The squared eigenfunction Ψ generates solutions of the adjoint linearized KdV equation. Let q and $Q = q+\delta q$ denote two solutions of the KdV equation. Then, as their difference δq goes to zero, it satisfies the linear equation

$$\partial_\tau \delta q - 6 \partial_y q\delta q + \partial_{yyy} \delta q = 0 \ , \qquad \text{(III.8)}$$

with formal adjoint

$$\partial_\tau \Psi - 6q \partial_y \Psi + \partial_{yyy} \Psi = 0 \qquad \text{(III.9)}$$

Let q denote a solution of KdV, and Ψ denote a solution of the pair (III.2). Then it immediately follows that Ψ satisfies the adjoint equation (III.9); thus, the coefficients Ψ_j are also solutions of the adjoint equation.

III. C. The μ Representation of KdV Waves

The squared eigenfunction system (III.2a,b) can be used to generate representations of N-phase watetrains. This construction begins with the observation that system (III.2a,b) admits a first integral [22]:

$$\frac{1}{2}\Psi\Psi_{yy} - \frac{1}{4}\Psi_y^2 - (q-\lambda)\Psi^2 = R^2(\lambda) \qquad \text{(III.10)}$$

To construct N-phase waves, we fix (2N+1) real constants $\vec{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{2N})$,

$$\lambda_0 < \lambda_1 < \ldots < \lambda_{2N}, \qquad \text{(III.11)}$$

and demand that the first integral $R^2(\lambda)$ be the polynomial

$$R^2(\lambda) = \prod_{k=0}^{2N} (\lambda-\lambda_k). \qquad \text{(III.12)}$$

This situation is achieved by seeking a solution $\Psi^{(N)}$ of the squared eigenfunction system which is polynomial in λ of degree N:

$$\Psi^{(N)}(y,\tau;\lambda) = \prod_{j=1}^{N} \left(\lambda-\mu_j(y,\tau)\right). \qquad \text{(III.13)}$$

Inserting this ansatz into (III.2) leads to the "μ-representation of KdV waves with N degrees of freedom":

$$q_N(y,\tau) = \Lambda - 2\sum_{j=1}^{N} \mu_j(y,\tau) \ , \tag{III.14a}$$

$$\psi^{(N)}(y,\tau;\lambda) = \prod_{j=1}^{N} (\lambda-\mu_j(y,\tau)) \ , \tag{III.14b}$$

where $\Lambda = \sum_{j=0}^{2N} \lambda_j$, and

where the $\mu_j(y,\tau)$ are constrained by

$$\lambda_0 \le \lambda_1 \le \mu_1(y,\tau) \le \lambda_2 \le \lambda_3 \le \mu_2(y,\tau) \le \lambda_4 \le \cdots \le \lambda_{2N-1} \le \mu_N(y,\tau) \le \lambda_{2N}, \tag{III.14c}$$

and satisfy the ordinary differential equations

$$\partial_y \mu_j = 2i \frac{R(\mu_j)}{\prod_{i\neq j} (\mu_j-\mu_i)} \tag{III.14d}$$

$$\partial_\tau \mu_j = -2i \left[2\left(\Lambda - 2\sum_{i\neq j} \mu_i\right)\right] \frac{R(\mu_j)}{\prod_{i\neq j} (\mu_j-\mu_i)} \ . \tag{III.14e}$$

When evaluated on q_N, the fundamental conservation law (III.3) has the μ-representation

$$\partial_\tau T + \partial_y X = 0 \ , \tag{III.14f}$$

$$T = \Psi = \frac{\psi^{(N)}(y,\tau;\lambda)}{R(\lambda)} = \frac{\prod_{j=1}^{N} (\lambda-\mu_j(y,\tau))}{\sqrt{\prod_{k=0}^{2N} (\lambda-\lambda_k)}} \tag{III.14g}$$

$$X = \left[6\Lambda - 12\left(\lambda + \sum_{j=1}^{N} \mu_j(y,\tau)\right)\right] \frac{\prod_{j=1}^{N} (\lambda-\mu_j(y,\tau))}{\sqrt{\prod_{k=0}^{2N} (\lambda-\lambda_k)}} - 2T_{yy} \quad \text{(III.1}$$

Formulas (III.14) summarize the "μ-representation of KdV waves with N-degrees of freedom". We now show that these exact solutions of KdV are "N phase wavetrains"; that is, they are solutions which are quasi-periodic in space and time which depend upon N phases.

III. D. The Θ Representation of KdV Waves

The wave q_N admits an equivalent representation which results once an Abel transformation is used to integrate the μ equations (III.14d,e). On the Riemann surface

$\mathcal{R} = [\lambda, R(\lambda) \equiv \sqrt{\prod_{0}^{2N} (\lambda - \lambda_k)}]$, one fixes a canonical set of a-b cycles (Figure 1). On this Riemann surface, we introduce the following objects: (i) a basis of holomorphic differentials,

Figure 1

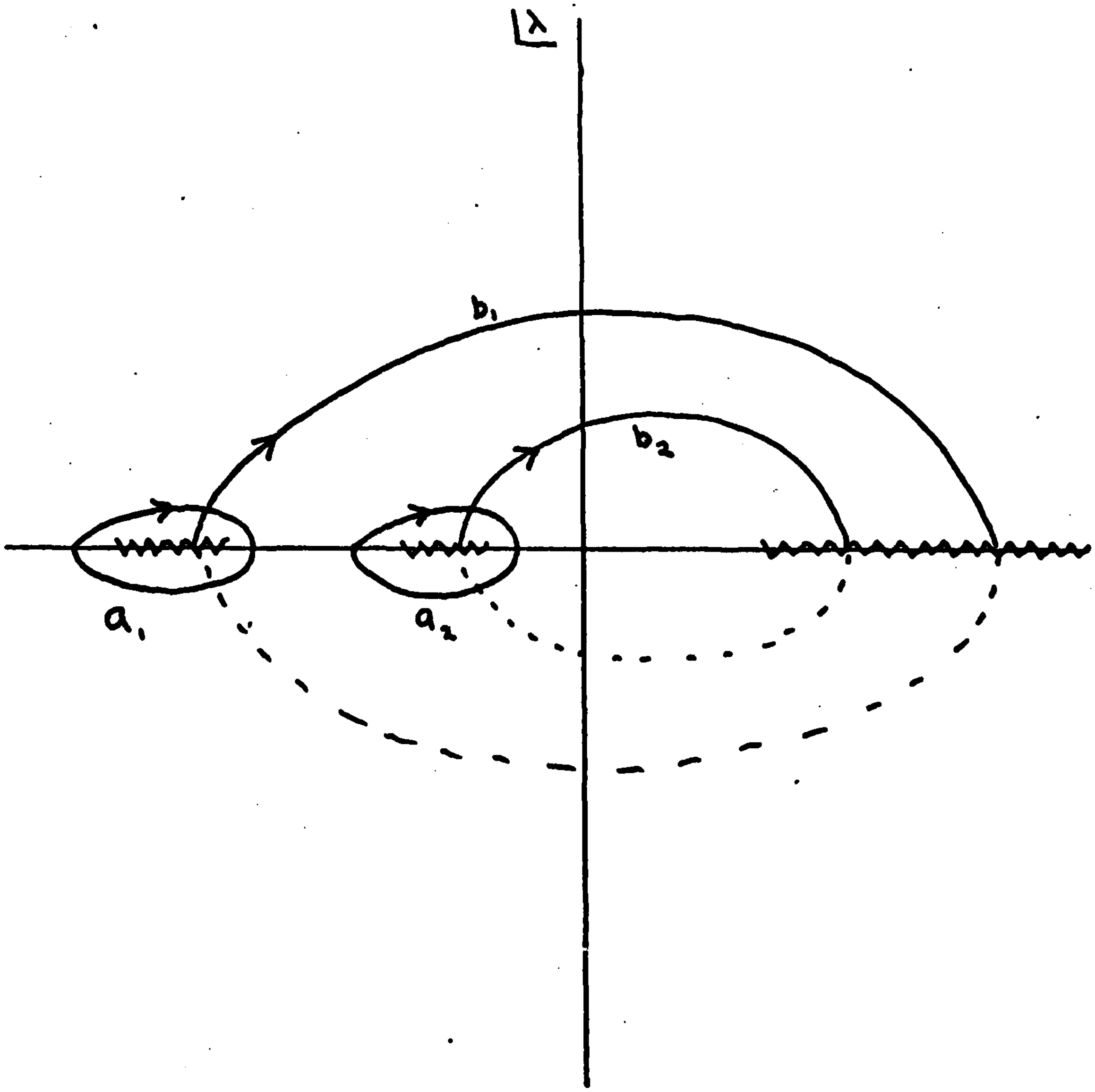

Figure 1

$$\psi_i \equiv \sum_{j=1}^{N} c_{ij}\, \lambda^{j-1} \frac{d\lambda}{R(\lambda)} \,, \tag{III.15a}$$

normalized by the condition

$$\oint_{a_i} \psi_j = \delta_{ij} \;; \tag{III.15b}$$

(ii) A symmetric matrix with positive definite imaginary part,

$$B_{ij} \equiv \oint_{b_i} \psi_j \;; \tag{III.15c}$$

(iii) Two differentials of the second kind

$$\Omega_1 \equiv \left[-\frac{1}{2} + \sum_{j=1}^{N} D_j\, \lambda^{j-1} \right] \frac{d\lambda}{R(\lambda)} \,,$$

$$\Omega_2 \equiv \left[-\frac{1}{2}\lambda^{N+1} + \frac{1}{4}\Lambda\, \lambda^{N} + \sum_{j=1}^{N} E_j\, \lambda^{j-1} \right] \frac{d\lambda}{R(\lambda)} \,, \tag{III.16a,b}$$

where the coefficient $\Lambda \equiv \sum_{k=0}^{2N} \lambda_k$, and the coefficients D_j and E_j are uniquely determined by the normalization conditions

$$\oint_{b_i} \Omega_j = 0$$

These quantities form the ingredients for a change of variables from $(\mu_1,\ldots,\mu_N) \rightarrow (\Theta_1,\ldots,\Theta_N)$:

$$\theta_i = (B^{-1})_{ij} \sum_{k=1}^{N} \int_{\hat{\mu}}^{\mu_k} \psi_j \qquad \text{(III.17*)}$$

If $\mu_j(y,\tau)$ satisfies the differential equations (III.14d,e), then the new variables $\theta_j(y,\tau)$ satisfy

$$\begin{aligned} \partial_y \theta_j &= \kappa_j \\ \partial_\tau \theta_j &= \omega_j \ , \end{aligned} \qquad \text{(II.18a,b)}$$

where the constants κ_j and ω_j are defined in terms of $\vec{\lambda}$:

$$\begin{aligned} \kappa_j &= \kappa_j(\vec{\lambda}) = -\oint_{a_j} \Omega_1 \\ \omega_j &= \omega_j(\vec{\lambda}) = -12 \oint_{a_j} \Omega_2 \ . \end{aligned} \qquad \text{(II.19a,b)}$$

Using theta functions, one can invert transformation (III.17) and give formulas for μ_j (and therefore q_N) in terms of $(\theta_j,\ldots,\theta_N)$:

$$q_N = \Lambda + \Gamma - 2\partial_{yy} \log \left[\Theta(z(\theta);B)\right] \ , \qquad \text{(III.20a)}$$

where the theta function is defined (for $z \in C^N$) by

$$\Theta(z;B) \equiv \sum_{m \in Z^N} \exp\{\pi i [2(m,z) + (m,Bm)]\} \ , \qquad \text{(III.20b)}$$

and $z(\Theta)$ denotes the linear map

$$z_j(0) = \sum_{j=1}^{N} \varepsilon \frac{B}{2\pi}^{(j)} \Theta_j + d_j \, ,$$

(III.20c)

where $B^{(j)}$ is the $j^{\underline{th}}$ column of the period matrix B

Finally, the constant Γ is given by

$$\Gamma \equiv -2 \sum_{j=1}^{N} \oint_{a_j} \lambda \, \psi_j \tag{III.20d}$$

and $\vec{d}$ denotes a real constant which plays no role in the following.

In this manner the theory of the exact KdV equation has generated a solution (III.20a) in the form

$$q_N = q_N(\Theta_1, \ldots, \Theta_N; \vec{\lambda})$$

where the (y,τ) dependence enters only through the phases,

$$\Theta_j(y,t) = \kappa_j y + \omega_j \tau + \dot{\Theta}_j \, ,$$

and where the wave form is 2π periodic in each individual phase. Thus, the constants κ_j and ω_j are enterpreted physically as spatial wave numbers and temporal frequencies.

This completes the summary of that material from the inverse spectral representation of exact KdV waves which we need for this paper. Now we return to the analysis of the sequence of problems (II.8).

IV. The Leading Order ($O(\varepsilon^{-1})$) Problem.

In this section we construct solutions of (II.8a). Fix $2N+1$ real constants $\vec{\lambda} = (\lambda_0 < \lambda_1 < \ldots < \lambda_{2N})$, and construct the B matrix and the differentials Ω_j as in (III.15) and (III.16). Define the wave vector $\kappa=\kappa(\vec{\lambda})$ and the freqeuncy vector $\omega = \omega(\vec{\lambda})$ by

$$\kappa_i(\vec{\lambda}) = -\oint_{a_i} \Omega_1$$

$$\omega_i(\vec{\lambda}) = -12 \oint_{a_i} \Omega_2 \quad . \tag{IV.1a,b}$$

For this $\kappa=\kappa(\vec{\lambda})$ and $\omega=\omega(\vec{\lambda})$, we seek $W:T^N(\text{N-Torus}) \to \mathbb{R}$ which satisfies the $O(\varepsilon^{-1})$ problem:

$$(\omega\cdot\nabla)W - 6W(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3 W = 0 \tag{IV.2}$$

Using material in Section III, we find solutions of (IV.2):

$$W = W(\Theta;\overset{\circ}{\Theta},\vec{\lambda}) = q_N(\Theta-\overset{\circ}{\Theta};\vec{\lambda}) \tag{IV.3}$$

Equivalently, we may use the μ-representation:

$$W = W(\Theta;\overset{\circ}{\Theta},\vec{\lambda}) = \Lambda - 2\sum_{j=1}^{N} \mu_j(\Theta-\overset{\circ}{\Theta};\vec{\lambda}) \quad , \tag{IV.4}$$

where the μ variables satisfy

$$(\kappa\cdot\nabla)\mu_j = -2i \frac{R(\mu_j)}{\prod_{i\neq j}(\mu_i-\mu_j)} \tag{IV.5a}$$

$$(\omega\cdot\nabla)\mu_j = -2i\left[2\Lambda - 2\sum_{i\neq j}\mu_j\right]\frac{R(\mu_j)}{\prod_{i\neq j}(\mu_j - \mu_i)} \qquad \text{(IV.5b)}$$

In this manner we generate a $3N+1$ real parameter family of solutions $W:T^N \to \mathbb{R}$ of (IV.2). N of the parameters, $\mathring{\theta}=(\mathring{\theta}_1\ldots,\mathring{\theta}_N)$, are trivial in that they merely center the N phases $\theta=(\theta_1\ldots,\theta_N)$. The remaining $2N+1$ parameters $\vec{\lambda} = (\lambda_0,\lambda_1,\ldots,\lambda_{2N})$ carry qualitative information about the wave. For example, they determine the wave numbers $\kappa=(\kappa_1,\ldots,\kappa_N)$ and the frequencies $\omega=(\omega_1,\ldots,\omega_N)$ by (IV.1). In addition, they determine the mean of the wave W:

$$\langle W\rangle = \frac{1}{(2\pi)^N}\int_{T^N} W(\theta)\, d^N\theta \,. \qquad \text{(IV.6)}$$

<u>Remark(i)</u> Physically, it seems more natural to co-ordinatize the N phase waves by the N spatial wave numbers κ, N temporal frequencies ω, and mean $\langle W\rangle$ rather than by the $2N+1$ parameters $\vec{\lambda}$. However, the $\vec{\lambda}$ co-ordinates are better understood mathematically.

<u>Remark (ii)</u> For the single phase $(N=1)$ case, equation (IV.2) is an ordinary differential equation of third order which is easy to analyse. One quickly shows that <u>all</u> of its 2π periodic solutions belong to the 4 parameter family (IV.3) with parameters $(\lambda_0,\lambda_1,\lambda_2,\mathring{\theta})$.

<u>Remark (iii)</u> For $N > 1$, I suspect that all solutions of (IV.2) which genuinely depend upon all N phases ($\frac{\partial W}{\partial \Theta_j} \not\equiv$ for any j) belong to family (IV.3). Assume there exists $W:T^N \to \mathbb{R}$ which solves (IV.2), depends genuinely upon all N phases, and does not belong to family (IV.3). Then $W(y,\tau) = W(\kappa_1 y+\omega_1\tau,\ldots,\kappa_N y+\omega_N\tau)$ is a solution of the KdV equation which is quasi-periodic in y and τ with exactly N frequencies, and yet is not an "N-gap potential" for inverse spectral theory. I think that no such solution exists.

<u>Remark (iv)</u> Notice that in this theory it is easy to shut off a phase, say Θ_j . One seeks a W which is independent of Θ_j. If κ_j and ω_j go to infinity, this situation is forced upon us. On the other hand, if κ_j and ω_j vanish (a soliton limit), $\partial/\partial\Theta_j$ is removed from equation (IV.2). In any case, such situations must be understood before a modulation theory sufficiently general to create and destroy phases can be developed.

V. Solvability Theory for the Linear Problems

Fix W, a solution of (IV.2) in family (IV.3). All of the $0(\varepsilon^j)$, $j \geq 0$, problems are of the form

$$LU + F = 0 , \tag{V.1}$$

with a prescribed inhomogeneity $F: T^N \to \mathbb{R}$. Here the linear operator L is defined in terms of W by

$$LU = (\omega\cdot\nabla)U - 6(\kappa\cdot\nabla)\ W\ U + (\kappa\cdot\nabla)^3\ U \tag{V.2}$$

(We work in the Hilbert space of functions $U: T^N \to \mathbb{R}$ which are square integrable over the torus T^N .)

For the solvability theory of (V.1), we need to understand $\eta(L)$, the null space of L, and $\eta(L^\dagger)$, the null space of the adjoint of L. Here the formal adjoint is given by

$$L^\dagger V = -\ (\omega\cdot\nabla)V + 6W(\kappa\cdot\nabla)V - (\kappa\cdot\nabla)^3 V . \tag{V.3}$$

We have the following fact concerning the null space $\eta(L^\dagger)$:

Theorem VI: (a) $\Psi^{(N)} \equiv \prod_{j=1}^{N} (\lambda-\mu_j)\ \varepsilon\ \eta(L^\dagger)\ \forall\ \lambda$

(b) $\Psi \equiv \dfrac{\Psi^{(N)}}{R(\lambda)} \approx \sum_{j=0}^{\infty} \Psi_j (2\lambda)^{-j-1/2}\ \varepsilon\ \eta(L^\dagger) \forall\ \lambda$

(V.4a,b,c)

(c) $\Psi_j\ \varepsilon\ \eta(L^\dagger)\ \forall\ j = 0,1,...$

The proof of this theorem follows immediately from the material around (III.14), together with the fact that $\mu_j(y,t)$ depends upon (y,t) only through the phases $\Theta_j(y,t) = \kappa_j y + \omega_j t$.

<u>Theorem V.2:</u> Formulas (V.4) generate only $(N+1)$ linearly independent members of $\eta(L^{\dagger})$. These may be represented as

$$\text{(a)} \quad \{\sigma_0, \sigma_1, \ldots, \sigma_N\}, \quad \text{where} \quad \prod_{j=1}^{N} (\lambda - \mu_j) = \sum_{j=0}^{N} \sigma_j \lambda^j ,$$

$$\text{(b)} \quad \{\Psi_0, \Psi_1, \ldots, \Psi_N\} \tag{V.5a,b}$$

<u>Theorem V.3:</u>

$$\text{(a)} \quad \frac{\partial W}{\partial \Theta_j} \in \eta(L) \quad \forall\ j=1,2,\ldots,N$$

$$\text{(b)} \quad 1 - 6\,(\kappa \cdot \nabla_\omega)\, W \in \eta(L) \tag{V.6a,b}$$

Using formulas (V.5), we have $(N+1)$ solvability conditions which are necessary if (V.1) is to have a solution. For $N=1$, simple analysis of the ordinary differential equation shows that these are actually necessary and sufficient. We have

<u>Theorem V.4</u> Let $N = 1$. Then

$$\text{(a)} \quad \eta(L^{\dagger}) = \text{span}\,\{1, W\}$$

$$\text{(b)} \quad \eta(L) = \text{span}\,\{W_\Theta,\ 1 - 6\,\kappa\, W_\omega\} \tag{V.7}$$

Thus, in the $N=1$ case, both $\eta(L)$ and $\eta(L^{\dagger})$ have dimension 2.

<u>Remark (i)</u> For $N > 1$, we suspect that $\eta(L)$ and $\eta(L^{\dagger})$ have dimension $N+1$. If so, $\eta(L^{\dagger}) = \text{span } \{\Psi_0,\ldots,\Psi_N\}$. We have not succeeded in proving this. For now, the solvability conditions

$$(\Psi_j, F) = 0 \text{ , for } j = 0,1,\ldots,N \qquad \text{(V.8)}$$

are necessary for $N > 1$; necessary and sufficient for $N = 1$.

VI The $O(\varepsilon^0)$ Problem

Armed with this solvability theory, we return to the sequence of linear problems (II.8b). Explicitly, the $O(\varepsilon^0)$ problem is

$$L\, U^{(0)} + \tilde{F}^{(0)} + \tilde{\tilde{F}}^{(0)} = 0 \,, \tag{VI,1a}$$

$$\tilde{F}^{(0)} = [W_t - 6WW_x + 3\Big((\kappa\cdot\nabla)^2 W_x + (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W$$

$$+ \beta f(W,\kappa\cdot\nabla W,\ldots)] \,, \tag{VI,1b}$$

$$\tilde{\tilde{F}}^{(0)} = [(\omega^{(1)}\cdot\nabla)W - 6W\,\kappa^{(1)}\cdot\nabla W + 3(\kappa\cdot\nabla)^2(\kappa^{(1)}\cdot\nabla)W] \,.$$

In the source, one does not have to worry about $\tilde{\tilde{F}}^{(0)}$ because for this part of the source, we have an explicit solution. Recall that W satisfies

$$(\omega\cdot\nabla)W - 6W(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3 W = 0.$$

Define $2N$ functions on the torus T^N by

$$\chi^{(\omega_j)} \equiv \frac{\partial W}{\partial \omega_j}, \tag{VI,2}$$

$$\chi^{(\kappa_j)} \equiv \frac{\partial W}{\partial \kappa_j} \,, \; j = 1,2,\ldots N.$$

Then, by differentiating the W equation, one finds

$$L\chi^{(\omega_j)} + \frac{\partial}{\partial \theta_j}\; W = 0 \tag{VI,3}$$

$$L\chi^{(\kappa_j)} - 6W\frac{\partial}{\partial \theta_j}\; W + 3(\kappa\cdot\nabla)^2\frac{\partial}{\partial \theta_j}\; W = 0$$

Using (VI,3) we may write $U^{(0)}$ as follows:

$$U^{(0)} = \sum_{j=1}^{N} [\omega_j^{(1)} \chi^{(\omega_j)} + \kappa_j^{(1)} \chi^{(\kappa_j')}] + \tilde{U}^{(0)} \qquad \text{(VI,4a)}$$

where

$$L\, \tilde{U}^{(0)} + \tilde{F}^{(0)} = 0 \qquad \text{(VI,4b)}$$

Thus, the solvability theory need only treat $\tilde{F}^{(0)}$.

Theorem (V,2) provides $N + 1$ necessary conditions for solvability,

$$(\Psi_j\ ,\ \tilde{F}^{(0)}) = 0 \quad \text{for} \quad j = (0,1,\ldots,N). \qquad \text{(VI,5)}$$

In this manner, we arrive at the modulation equations which must be satisfied:

$$\partial_t \kappa_j = \partial_x \omega_j \quad , \quad j = 1,2,\ldots,N$$

$$\left(\Psi_j,\ W_t - 6WW_x + 3\,[(\kappa\cdot\nabla)^2 W_x + (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W] + \beta f(W,\kappa\cdot\nabla W,\ldots)\right) = 0, j=0,1,\ldots,N \qquad \text{(VI,6)}$$

Equations (VI,6) are the main result of this section. They provide a system of $2N+1$ first order partial differential equations for the $2N+1$ parameters $\vec{\lambda} = (\lambda_0,\ldots,\lambda_{2N})$. The first N equations result from consistency of the ansatz; the last $N+1$ from necessary solvability conditions. Equations (VI,6) provide a closed system which depends only upon $\vec{\lambda}$, $\vec{\lambda}_x$, $\vec{\lambda}_t$. In the next section, we place this system in manageable form.

If the null space is exactly $N+1$ dimensional (as we know for $N=1$ and suspect for $N > 1$), these solvability

conditions are necessary and sufficient to ensure the existence of $U^{(1)} : T^N \to \mathbb{R}$. At this stage, one can proceed in two different directions. (i) One can continue to generate higher order terms $U^{(j)}$ in the expansion of U^ε The corrections to the frequencies ω_j^ε, the wave numbers κ_j^ε, and the mean $\langle U^\varepsilon \rangle$ will provide sufficient freedom to ensure solvability at each stage. In this manner, a formal asymptotic expansion of the form
$U^\varepsilon \sim W_N + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \dots$ can be constructed.
(ii) Alternatively, one can truncate the expansion at $W_N \left(\frac{\sigma(x,t)}{\varepsilon} ; \vec{\lambda}(x,t)\right)$ where the $\vec{\lambda}(x,t)$ satisfy the modulation equations (VI,6) and prove that $U^\varepsilon - W_N$ is $O(\varepsilon)$ with some uniformity. The second direction is the most important.

VII. Connection between the Modulation Equation and Averaged Conservation Laws

The modulation equations

$$\partial_t \kappa_j = \partial_x \omega_j \,, \qquad j = 1,2,\ldots,N$$

(VII.1a,b)

$$\left(\Psi_j,\; W_t - 6WW_x + 3[(\kappa\cdot\nabla)^2 W_x + (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W] + \beta f(W, \kappa\cdot\nabla W, \ldots)\right) = 0 \,, \qquad j = 0,1,2,\ldots,N,$$

although a closed system for $\vec{\lambda}(x,t)$, appear to be a complicated system of nonlinear partial differential equations. In [3] we show that, even in the presence of an external perturbation *f*, these modulation equations are actually very tractable provided (VII.1b) can be replaced by N+1 averaged conservation laws. In this section we derive the validity of this replacement for N = 1,2. (This is sufficient to treat 2 phase waves.)

VII.A. Averaged Conservation Laws

One approach to deriving modulation equations is to average conservation laws. One assumes that the exact equation,

$$U_t - 6UU_x + \varepsilon^2 U_{xxx} + \beta f(U, \varepsilon U_x, \varepsilon^2 U_{xx}, \ldots) = 0, \qquad \text{(VII.1)}$$

has a solution of the form

$$U \sim W_N\left(\frac{\theta_1(x,t)}{\varepsilon}, \ldots, \frac{\theta_N(x,t)}{\varepsilon}; \vec{\lambda}(x,t)\right) + O(\varepsilon) \,, \qquad \text{(VII.2)}$$

which is 2π periodic in each θ_j. In addition, it has conservation laws of the form

$$\partial_t T(U, \varepsilon U_x, \ldots) + \partial_x X(U, \varepsilon U_x, \ldots) + \beta G(U, \varepsilon U_x, \ldots) = 0 \, , \quad \text{(VII.3)}$$

for any solution of (VII.1). In particular, evaluating on solution (VII.2), conservation law (VII.3) takes the form ($\partial_t \to \frac{1}{\varepsilon}\, \omega \cdot \nabla + \partial_t$, etc.).

$$\begin{aligned} &\frac{1}{\varepsilon}\left[(\omega \cdot \nabla)\, T + (\kappa \cdot \nabla) X\right] \\ &+ \frac{\partial}{\partial t} T(W_N, (\kappa \cdot \nabla) W_N, \ldots) + \frac{\partial}{\partial x} X(W_N, (\kappa \cdot \nabla) W_N, \ldots) + \beta G(W_N, (\kappa \cdot \nabla) W_N, \ldots \\ &+ \ldots = 0 \end{aligned} \quad \text{(VII.4}$$

When averaged over T^N, the $O(\varepsilon^{-1})$ term averages to zero, and one is left with the averaged conservation law

$$\begin{aligned} &\partial_t \langle T(W_N, (\kappa \cdot \nabla) W_N, \ldots)\rangle + \partial_x \langle X(W_N, (\kappa \cdot \nabla) W_N, \ldots)\rangle \\ &+ \beta \langle G(W_N, (\kappa \cdot \nabla) W_N, \ldots)\rangle = 0 \end{aligned} \quad \text{(VII.5}$$

This is one equation among 2N+1 unknowns.

Each KdV conservation law will lead to an averaged conservation law of the form (VII.5). Indeed, consider j^{th} KdV density as generated by the recursion relation (III.5), $\Psi_j = \Psi_j(U, \varepsilon\partial_x U, \ldots, (\varepsilon\partial_x)^{k_j} U)$. Here k_j is the order of the highest derivative of U in Ψ_j . We compute:

$$\partial_t \Psi_j = \sum_{\ell=0}^{k_j} \Psi_{j,\ell}\, \partial_t (\varepsilon\partial_x)^{\ell} U$$

$$= \sum_{\ell=0}^{k_j} \Psi_{j,\ell} (\varepsilon\partial_x)^{\ell} [6UU_x - \varepsilon^2 U_{xx} - \beta f(U, \varepsilon U_x, \ldots)]$$

$$= -\partial_x X_j - \beta \sum_{\ell=0}^{k_j} \Psi_{j,\ell} (\varepsilon\partial_x)^{\ell}\, f(U, \varepsilon U_x, \ldots)\ ,$$

where $\Psi_{j,\ell}$ denotes a partial derivative. Thus, we obtain the perturbed conservation law

$$\partial_t \Psi_j + \partial_x X_j = -\beta \sum_{\ell=0}^{k_j} \Psi_{j,\ell} (\varepsilon\partial_x)^{\ell}\, f(U, \varepsilon U_x, \ldots)\ ,$$

with the right hand side explicitly given in terms of the j^{th} density and the external perturbation f. Evaluating this conservation law on the wave form (VII.2), and averaging over the torus T^N yields the perturbed avaraged conservation law

$$\partial_t \langle \Psi_j \rangle + \partial_x \langle 6(W\Psi_j - \Psi_{j+1}) \rangle + \beta \langle G_j \rangle, \tag{VII.6a}$$

where

$$\langle G_j \rangle \equiv \sum_{\ell=0}^{k_j} \left(\Psi_{j,\ell}, (\kappa\cdot\nabla)^{\ell}\, f(W, \kappa\cdot\nabla W, \ldots) \right), \tag{VII.6b}$$

and the densities Ψ_j are evaluated on the N phase wave form W. For later use, we list the first three:

$$\partial_t \langle W \rangle - \partial_x \langle 3W^2 \rangle + \beta \langle f(W, (\kappa\cdot\nabla)W, \ldots) \rangle = 0$$

$$\partial_t \langle \frac{W^2}{2} \rangle - \partial_x \langle 2W^2 + \frac{3}{2}\, (\kappa\cdot\nabla W)^2 \rangle + \beta \langle W\, f(W, \kappa\cdot\nabla W, \ldots) \rangle = 0 \tag{VII.a,b,c}$$

$$\partial_t \langle W^3 + \frac{1}{2}\, (\kappa\cdot\nabla W)^2 \rangle - \partial_x \langle \frac{9}{2}\, W^4 + 12W(\kappa\cdot\nabla W)^2 + \frac{3}{2}\, ((\kappa\cdot\nabla)^2 W)^2$$

$$+ \beta \langle 3W^2 f + (\kappa\cdot\nabla)W\ (\kappa\cdot\nabla) f \rangle = 0$$

VII.B. Connection between Averaged Conservation Laws and Null Space Modulation Equations

Consider the null space equations (VI.5),

$$(\Psi_j,\tilde{F}^{(0)}) = 0\,, \qquad j = 0,1,2,\ldots,N,$$

$$\tilde{F}^{(0)} = W_t - 6WW_x + 3\big((\kappa\cdot\nabla)^2W_x + (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W\big) + \beta f(W,(\kappa\cdot\nabla)W,\ldots) \qquad \text{(VII.8a,b)}$$

We show the first three of these are averaged conservation laws; indeed, they become (VII.6) for j = 1,2,3. Since $\Psi_0 = 1$, the first is immediate:

$$0 = (\Psi_0,\tilde{F}^{(0)}) = (1,\ \tilde{F}^{(0)})$$
$$= \partial_t\langle W\rangle - \partial_x\langle 3W^2\rangle + \beta\langle f\rangle$$

The next is almost as simple:

$$0 = (\Psi_1,\tilde{F}^{(0)}) = (W,\tilde{F}^{(0)})$$
$$= (W,W_t) - (W,6WW_x) + 3(W,(\kappa\cdot\nabla)^2W_x + (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W) + \beta(W,f)$$
$$= \frac{1}{2}\,\partial_t\langle W^2\rangle - 2\,\partial_x\langle W^3\rangle - 3\big(\kappa\cdot\nabla W,\partial_x(\kappa\cdot\nabla)W\big) + \beta(W,f)$$
$$= \partial_t\langle \frac{W^2}{2}\rangle - \partial_x\langle 2W^3 + \frac{3}{2}\,(\kappa\cdot\nabla W)^2\rangle + \beta\langle W\,f\rangle.$$

The verification that the third null space equation yields the third averaged conservation law is more tedious. It uses an extra ingredient -- $\partial_t\kappa_j = \partial_x\omega_j$. This calculation begins as follows:

$$0 = 2(\Psi_2, \tilde{F}^{(0)}) = (3W^2 - (\kappa\cdot\nabla)^2 W, \tilde{F}^{(0)})$$

$$= (3W^2 - (\kappa\cdot\nabla)^2 W,\ W_t - 3(W^2)_x + 3(\kappa\cdot\nabla)^2 W_x + 3(\kappa\cdot\nabla)(\kappa_x\cdot\nabla) W + \beta f)$$

$$= (3W^2, W_t) - ((\kappa\cdot\nabla)^2 W, W_t) + 9(W^2, (W^2)_x) + 3((\kappa\cdot\nabla)^2 W, (W^2)_x)$$

$$+ 9(W^2, (\kappa\cdot\nabla)^2 W_x) - 3((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)^2 W_x)$$

$$+ 9(W^2, (\kappa\cdot\nabla)(\kappa_x\cdot\nabla) W) - 3((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)(\kappa_x\cdot\nabla) W)$$

$$+ 3\beta(W^2, f) - \beta((\kappa\cdot\nabla)^2 W, f)$$

$$= \partial_t \langle W^3 \rangle + \big((\kappa\cdot\nabla)W, (\kappa\cdot\nabla)W_t\big) - \frac{9}{2}\partial_x \langle W^4 \rangle - 3\big((\kappa\cdot\nabla)W, (\kappa\cdot\nabla)(W^2)_x\big)$$

$$- 9((\kappa\cdot\nabla)W^2, (\kappa\cdot\nabla)W_x) - 3((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)^2 W_x) - 9((\kappa\cdot\nabla)W^2, (\kappa_x\cdot\nabla)W)$$

$$- 3((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W) + \beta\langle f[3W^2 - (\kappa\cdot\nabla)^2 W] \rangle$$

$$= \partial_t \langle W^3 \rangle + \partial_t \left\langle \frac{[(\kappa\cdot\nabla)W]^2}{2} \right\rangle - ((\kappa\cdot\nabla)W, (\kappa_t\cdot\nabla)W) - \partial_x \left\langle \frac{9}{2} W^4 \right\rangle$$

$$- 3\big((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)^2 W_x\big) - 3\big((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W\big)$$

$$- 3\big((\kappa\cdot\nabla)W, (\kappa\cdot\nabla)(W^2)_x\big) - 9\big((\kappa\cdot\nabla)W_x, (\kappa\cdot\nabla)W^2\big)$$

$$- 9\big((\kappa\cdot\nabla)W^2, (\kappa_x\cdot\nabla)W\big) + \beta\langle f[3W^2 - (\kappa\cdot\nabla)^2 W] \rangle$$

Continuing,

$$0 = \partial_t\langle W^3 + \frac{1}{2}(\kappa\cdot\nabla W)^2\rangle - \partial_x\langle \frac{9}{2} W^4\rangle + \beta\langle f[3W^2-(\kappa\cdot\nabla)^2 W]\rangle$$

$$- \Big((\kappa\cdot\nabla)W, (\kappa_t\cdot\nabla)W\Big) - \partial_x\langle \frac{3}{2}[(\kappa\cdot\nabla)^2 W]^2\rangle$$

$$+ 3\Big((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W\Big)$$

$$- \partial_x\langle 12W(\kappa\cdot\nabla W)^2\rangle + 12\Big(W_x, (\kappa\cdot\nabla W)^2\Big) + 24\Big(W, [(\kappa\cdot\nabla)W]\ \kappa\cdot\nabla W_x\Big)$$

$$+ 24\Big(W, [(\kappa\cdot\nabla)W](\kappa_x\cdot\nabla)W\Big) - 6\Big((\kappa\cdot\nabla)W, W(\kappa\cdot\nabla)W_x\Big) - 6\Big([(\kappa\cdot\nabla)W]^2, W_x\Big)$$

$$- 18\Big((\kappa\cdot\nabla)W_x, W(\kappa\cdot\nabla)W\Big) - 18\Big(W(\kappa\cdot\nabla)W, (\kappa_x\cdot\nabla)W\Big) .$$

Therefore,

$$\partial_t\langle W^3 + \frac{1}{2}(\kappa\cdot\nabla W)^2\rangle - \partial_x\langle \frac{9}{2} W^4 + 12W[\kappa\cdot\nabla W]^2$$

$$+ \frac{3}{2}[(\kappa\cdot\nabla)^2 W]^2\rangle + \beta\langle f[3W^2-(\kappa\cdot\nabla)^2 W]\rangle$$

$$= \Big((\kappa\cdot\nabla)W, (\kappa_t\cdot\nabla)W\Big) - 3\Big((\kappa\cdot\nabla)^2 W, (\kappa\cdot\nabla)(\kappa_x\cdot\nabla)W\Big)$$

$$- \Big((\kappa\cdot\nabla)W,\ 6W_x(\kappa\cdot\nabla)W + 6W(\kappa_x\cdot\nabla)W\Big) \qquad \text{(VII.9)}$$

$$= \Big((\kappa\cdot\nabla)W, (\kappa_t\cdot\nabla)W + 3(\kappa\cdot\nabla)^2(\kappa_x\cdot\nabla)W$$

$$- 6W_x(\kappa\cdot\nabla)W - 6W(\kappa_x\cdot\nabla)W\Big) .$$

Thus, the third null space condition will yield the third averaged conservation law provided the expression

$$\Big((\kappa\cdot\nabla)W, (\kappa_t\cdot\nabla)W + 3(\kappa\cdot\nabla)^2(\kappa_x\cdot\nabla)W - 6W_x(\kappa\cdot\nabla)W - 6W(\kappa_x\cdot\nabla)W\Big) \qquad \text{(VII.10)}$$

vanishes. To reach this point, we merely integrated the null space condition by parts several times. Now we employ the first N modulation equations

$$\kappa_t = \omega_x$$

to replace expression (VII.10) by

$$\big((\kappa\cdot\nabla)W, (\omega_x\cdot\nabla)W + 3(\kappa\cdot\nabla)^2(\kappa_x\cdot\nabla)W - 6W_x(\kappa\cdot\nabla)W - 6W(\kappa_x\cdot\nabla)W\big) \tag{VII.10'}$$

To show this expression vanishes, we use the equation satisfied by the N phase wave :

$$(\omega\cdot\nabla)W - 6W(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3W = 0 .$$

Differentiating this equation with respect to x yields

$$(\omega\cdot\nabla)W_x - 6W(\kappa\cdot\nabla)W_x - 6W_x(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3W$$
$$- (\omega_x\cdot\nabla)W - 6W(\kappa_x\cdot\nabla)W + 3(\kappa\cdot\nabla)^2(\kappa_x\cdot\nabla)W = 0 .$$

Finally, take the inner product with $(\kappa\cdot\nabla)W$:

$$\big((\kappa\cdot\nabla)W, (\omega\cdot\nabla)W_x - 6W(\kappa\cdot\nabla)W_x - 6W_x(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3W_x$$
$$+ (\omega_x\cdot\nabla)W - 6W(\kappa_x\cdot\nabla)W + 3(\kappa\cdot\nabla)^2(\kappa_x\cdot\nabla)W\big) = 0$$

That is,

$$\big((\kappa\cdot\nabla)W_x, (\omega\cdot\nabla)W - 6W(\kappa\cdot\nabla)W + (\kappa\cdot\nabla)^3W\big)$$
$$+ \big((\kappa\cdot\nabla)W, (\omega_x\cdot\nabla)W + 3(\kappa\cdot\nabla)^2(\kappa_x\cdot\nabla)W - 6W_x(\kappa\cdot\nabla)W - 6W(\kappa_x\cdot\nabla)W\big) = 0$$

The first term vanishes by the fact that W solves the N phase equation; thus, we arrive at the vanishing of (VII.10')!

In summary, we have shown that (i) the first two null space equations imply the first two averaged conservation laws, and that (ii) the third null space equation, together with the consistency conditions $\kappa_t = \omega_x$, imply the third averaged conservation law.

Remark (i). Presumably, the j^{th} null space equation, together with the consistency conditions, implies the j^{th} averaged conservation law. To prove this statement, one needs a more abstract argument than the explicit calculation described above. As yet, we have not succeeded.

Remark (ii). We have completed a first step toward this argument. Consider $(\delta H_j/\delta W, W_t)$. Using the identity [31]

$$H_j = \frac{1}{2j+1} \left\langle \frac{\delta H_{j+1}}{\delta W} \right\rangle$$

we obtain

$$\frac{\partial}{\partial t} \left\langle \frac{1}{2j+1} \frac{\delta H_{j+1}}{\delta W} \right\rangle = \frac{\partial}{\partial t} H_j$$
$$= \left(\frac{\partial H_j}{\partial W}, W_t\right) + \frac{\partial H_j}{\partial \kappa} \kappa_t \;;$$

where $H_j = H_j(W;k)$. Thus,

$$\left(\frac{\delta H_j}{\delta W}, W_t\right) = \frac{1}{2j+1} \frac{\partial}{\partial t} \left\langle \left.\frac{\delta H_{j+1}}{\delta q}\right|_W \right\rangle - \frac{\partial H_j}{\partial \kappa} \kappa_t \qquad \text{(VII.11)}$$

Equivalently, it may be better to use a formula from [ ],

$$\frac{\delta \Psi}{\delta W} = \Psi_\lambda + \frac{d}{dx} \left[-\frac{\Psi_\lambda}{2\Psi} \frac{\delta \Psi'}{\delta W} + \frac{\Psi'_\lambda}{2\Psi} \frac{\delta \Psi}{\delta W} \right] , \qquad \text{(VII.12)}$$

where $\Psi_\lambda = \partial_\lambda \Psi$, $\Psi' = \partial_x \Psi$.

VIII. An Invariant Form of the Modulation Equations

In this section, we assume the correct modulation equations are of the form

$$\partial_t \kappa_j = \partial_x \omega_j \qquad j = 1,2,\ldots,N$$

$$\partial_t \langle \Psi_j \rangle + \partial_x \langle \chi_j \rangle + \beta \langle G_j \rangle \qquad j = 1,2,\ldots,N+1 \tag{VIII.1a,b}$$

We summarize some results of [3] in order to emphasize that these modulation equations are indeed tractible.

In [3], we establish the following

Theorem The modulation equations (VIII.1) admit an equivalent representation in terms of the differentials Ω_1, Ω_2 :

$$\partial_t \Omega_1 - 12\, \partial_x \Omega_2 + \beta \, dF = 0 \quad , \tag{VIII.2}$$

where F is a meromorphic function on the Riemann surface R of the form

$$F(\lambda) = \frac{A(\lambda)}{R(\lambda)} \, , \quad A(\lambda) = \sum_{j=0}^{N} \alpha_j \, \lambda^j \; . \tag{VIII.3a}$$

The coefficients α_j are fixed in terms of $\langle G_j \rangle$ by the linear system

$$\sum_{k=0}^{j} \rho_{j-k} \left[(N-k)\alpha_{N-k} - \frac{1}{2} \sum_{m=0}^{k} \left(\sum_{\ell=0}^{2N} \lambda_\ell^{k-m} \right) \alpha_{N-m} \right] = \frac{1}{2^{j+1}} \langle G_{j+1} \rangle, \; j=0,1,2,\ldots \tag{VIII.3b}$$

Here

$$\frac{-2}{\sqrt{\prod_{\ell=0}^{2N} (1-\lambda_\ell \xi^2)}} = \sum_{k=0}^{\infty} \rho_k \, \xi^{2k} \, .$$

Representation (VIII.2) is fundamental. It contains alternative representations and quickly shows they are equivalent. For example, the most useful mathematical form is an immediate

Corollary (Riemann Invariant Form of the Modulation Equations).

By evaluating the invariant representation $\partial_t \Omega_1 - \partial_x \Omega_2 + \beta dF=0$ at the branch points, one obtains

$$\partial_t \lambda_\ell + \left[s^{(\ell)}(\vec{\lambda}) \right] \partial_x \lambda_\ell = \beta \frac{A(\lambda_\ell)}{\sum_{j=1}^{N+1} D_j \lambda_\ell^{j-1}} \, , \; \ell=0,1,\ldots,2N, \tag{VIII.4a}$$

where the ℓ^{th} characteristic speed $s^{(\ell)}(\vec{\lambda})$ is given by

$$s^{(\ell)}(\vec{\lambda}) \equiv \frac{-12 \sum_{j=1}^{N+2} E_j \, \lambda_\ell^{j-1}}{\sum_{j=1}^{N+1} D_j \lambda_\ell^{j-1}} \quad . \qquad \text{(VIII.4b)}$$

A fully nonlinear modulation theory cannot be simpler than the Riemann invariant form (VIII.4a). We emphasize the "internal perturbations" described in the introduction cause the parameters to modulate with the characteristic speeds $s^{(\ell)}(\vec{\lambda})$. The external perturbation f provides the right hand side of the modulation equations.

IX. The Weak Limit as a Measure

Recently, in some very interesting mathematical work [19,20,10] weak limits of solutions of nonlinear pde's have been described by a measure. When the nonlinear pde is dissipation dominated, as in Burgers' equation, the measure is simply a Dirac measure supported on the shock. When oscillations persist as in the small dispersion limit of KdV, the weak limit is more interesting. As yet, the measure has not been rigorously characterized.

Our purpose in this final section is to calculate, with heuristic reasoning, the measure which describes the weak limit as $\varepsilon \to 0$ of problem (II.1a), in a region of space-time where the solution is described by a modulating N-phase wave. *Since modulation theory constructs the* solution for small but finite ε, our representation certainly contains enough information to calculate the measure very explicitly.

Consider the solution of

$$U_t - 6UU_x + \varepsilon^2 U_{xxx} + \beta\, f(U, \varepsilon U_x, \ldots) = 0 \,, \qquad \text{(IX.1)}$$

as constructed in this paper. Namely, in a region S of space-time, let U^ε be the modulating N phase wave

$$U^{\varepsilon}(x,t) = W_N\left(\frac{\theta_1(x,t)}{}, \ldots, \frac{\theta_N(x,t)}{t} ; \vec{\lambda}(x,t)\right)$$

$$+ \varepsilon\, U^{(1)}\left(\frac{\vec{\theta}}{\varepsilon} ; x,t\right) + \ldots, \qquad \text{(IX.2)}$$

with the modulation of the parameters $\vec{\lambda}(x,t)$ described by

$$\Omega = dF \qquad \text{(IX.3)}$$

Fix some t and a spatial interval I such that $(t,I) \in S$; further, let $\phi: \mathbb{R} \to \mathbb{R}$ denote any test function with support within I. Fix $f: \mathbb{R} \to \mathbb{R}$ and consider

$$\left(\phi,\ f(U^{(\varepsilon)}(.,t))\right) = \int_{-\infty}^{\infty} \phi(x,t)\ f(U^{(\varepsilon)}(x,t))\, dx ,$$

which one must consider in order to describe the weak limit $f(U^{\varepsilon}(.,t))$ as $\varepsilon \downarrow 0$. We compute:

$$\begin{aligned}
\left(\phi, f(U^{(\varepsilon)}(.,t))\right) &= \int_{-\infty}^{\infty} \phi(x,t)\ f(U^{(\varepsilon)}(x,t))\, dx \\
&= \sum_{i=-\infty}^{\infty} \int_{x_i-\Delta/2}^{x_i+\Delta/2} \phi(x,t)\ f(U^{(\varepsilon)}(x,t))\, dx \\
&\simeq \sum_{i=-\infty}^{\infty} \int_{x_i-\Delta/2}^{x_i+\Delta/2} \phi(x,t)\ f\left[W_N\left(\frac{\theta(t,x)}{\varepsilon} ; \vec{\lambda}(t,x)\right)\right] dx \\
&\simeq \sum_{i=-\infty}^{\infty} \phi(x_i,t) \int_{x_i-\Delta/2}^{x_i+\Delta/2} f\left[W_N\left(\frac{\theta(t,x)}{\varepsilon} ; \vec{\lambda}(t,x_i)\right)\right] dx
\end{aligned}$$

$(\Delta \text{ tiny})$

$$\sum_{i=-\infty}^{\infty} \phi(x_i,t)\ \Delta \left\{\frac{1}{\Delta} \int_{x_i-\Delta/2}^{x_i+\Delta/2} f\left[W_N\left(\frac{\vec{\kappa}(x_i,t)x-\vec{\beta}(x_i,t)}{\varepsilon}\right);\ \vec{\lambda}(t,x_i)\ \right]\right\}$$

$$= \sum_{i=-\infty}^{\infty} \phi(x_i,t)\ \Delta \left\{\frac{\varepsilon}{\Delta} \int_{-\Delta/2\varepsilon}^{\Delta/2\varepsilon} f[W_N(\vec{\kappa}y - \vec{\beta};\ \vec{\lambda}(t,x_1))]\ dy\right\}$$

$$\simeq \sum_{i=-\infty}^{\infty} \phi(x_i,t)\ \Delta \left\{\frac{1}{(2\pi)^N} \int_{T^N} f[W_N(\theta_1,\ldots,\theta_N;\ \vec{\lambda}(x_i,t)]\, d^N\theta\right\}$$

$$(\varepsilon \to 0 \quad \text{and ergodicity})$$

$$\simeq \int_{-\infty}^{\infty} \phi(x,t)\ \langle f(W_N;\ \vec{\lambda}(x,t))\rangle\ dx \qquad (\Delta \to 0)$$

Thus, we compute

$$\lim_{\varepsilon \downarrow 0} \left(\phi, f(U^\varepsilon(.,t))\right) = \left(\phi, \langle f\rangle\right)\ \forall\ \phi\ ;$$

that is,

$$f(U^\varepsilon(x,t)) \xrightarrow[\varepsilon\to 0]{} \langle f(W_N(\cdot;\ \vec{\lambda}(x,t))\rangle \tag{IX.4}$$

$$= \frac{1}{(2\pi)^N} \int_{T^N} f[W_N(\theta;\ \vec{\lambda}(x,t))]\ d^N\theta.$$

The measure itself can be characterized very explicitly by using the μ-coordinates for the torus (see VII.18):

$$f(U^{\epsilon}(x,t)) \xrightarrow[\epsilon\to 0]{} \frac{1}{V} \oint_{b_1} \cdots \oint_{b_N} f[\Lambda(x,t) - \sum_{j=1}^{N} \mu_i] \cdot \frac{\prod_{i>1} (\mu_i - \mu_j)}{\prod_{i=1}^{N} |R(\mu_i; \vec{\lambda}(x,t))|} \, d\mu_1 \wedge \ldots d\mu_N \tag{IX.5}$$

(By translation, we can remove the function $\Lambda(x,t)$ from the argument of f.)

Formula (IX.5) is the main result of this section. It shows that the weak limit as $\epsilon \to 0$ can be characterized by a measure, and gives an explicit formula for the measure. Notice that the (x,t) dependence of the measure is through $\vec{\lambda}(x,t)$ which satisfies $\Omega = dF$. This is precisely the measure used in [1].

Remark. Our calculation is limited to the region S where the solution is an N phase wave. It is not uniformly applicable. However, we believe it indicates that the general weak limit, which would be valid for all space, should be characterized by the quadratic variational problem of [8,9].

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